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Journal of Sound and Vibration 295 (2006) 1099–1104
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Analytical approximations of the period of a generalized nonlinear van der Pol oscillator

Igor V. Andrianov[a,*], Wim T. van Horssen[b]

[a]Institute of General Mechanics, RWTH Aachen, Templergraben 64, Aachen, D-52062, Germany
[b]Delft Institute of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

Received 3 October 2005; received in revised form 10 February 2006; accepted 14 February 2006
Available online 4 April 2006

Abstract

In this paper analytical approximations for the period of a generalized nonlinear van der Pol equation will be obtained by using various asymptotic methods.

1. Introduction

In this paper a generalized nonlinear van der Pol equation will be studied. The following generalized van der Pol equation:

$$\ddot{x} + x^{(2m+1)/(2n+1)} = \varepsilon(1 - x^2)\dot{x}, \tag{1}$$

where $m, n \in \mathbb{N}$ and $0 \leqslant \varepsilon \ll 1$, has already been studied by Waluya and van Horssen by using a perturbation method based on integrating factors [1]. Hu and Xiong [2], Mickens et al. [3,4] also studied this equation by applying the generalized harmonic balance method. It is also possible to apply the saw-tooth approach [5] to analyze Eq. (1).

Oddness of both the numerator $(2m+1)$ and the denominator $(2n+1)$ of the exponent in Eq. (1) is important. If one of the parts in this ratio is even then Eq. (1) is not an oscillator equation.

In Ref. [6] it is proposed to modify Eq. (1) in the following way, which enables one to consider a more general class of oscillators:

$$\ddot{x} + \mathrm{sgn}(x)|x|^{\alpha} = \varepsilon(1 - x^2)\dot{x}, \quad \alpha > 0, \tag{2}$$

*Corresponding author. Tel.: +49 221 7089888; fax: +49 221 7006780.
E-mail addresses: igor_andrianov@hotmail.com (I.V. Andrianov), w.t.vanhorssen@ewi.tudelft.nl (W.T. van Horssen).
URL: http://andrianov.freeservers.com, http://ta.twi.tudelft.nl/dv/staff/Horssen/.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.02.006

where

$$\operatorname{sgn}(x) = \begin{cases} +1 & \text{for } x>0, \\ -1 & \text{for } x<0. \end{cases} \tag{3}$$

For $\alpha = (2m+1)/(2n+1)$ oscillator equation (2) is of course identical to Eq. (1). But the generalized form of Eq. (2) allows the exponent α to take any positive real value (such as odd, even, rational or irrational, and so on).

In order to get more insight in the period(s) of the periodic solution(s) of the generalized van der Pol equation (2) three cases will be considered: $\alpha \downarrow 0$, $\alpha \to \infty$, and $\alpha \to 1$. The parameter ε is assumed to be small, that is, $0<\varepsilon \ll 1$.

2. Integrating factor solution

By using an integrating factor approach Waluya and van Horssen [1] constructed asymptotic approximations of the periodic solutions and their periods for Eq. (1). In Ref. [1] as parameter of the asymptotic investigation ε has been used. A straightforward analysis shows that the results of Ref. [1] can be generalized to Eq. (2). Then one obtains as approximation for the period of the periodic solution that (for $\varepsilon \downarrow 0$ and for a fixed α with $0<\alpha<\infty$):

$$T(\alpha) = 2\sqrt{2}\sqrt{1+\alpha}A^{0.5(1-\alpha)} \int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^{1+\alpha}}} + \mathcal{O}(\varepsilon), \tag{4}$$

$$A = \left(\frac{J_1(\alpha)}{J_2(\alpha)}\right)^{1/2}, \tag{5}$$

where

$$J_1(\alpha) = \int_0^1 \sqrt{1-u^{1+\alpha}}\,\mathrm{d}u, \tag{6}$$

$$J_2(\alpha) = \int_0^1 u^2\sqrt{1-u^{1+\alpha}}\,\mathrm{d}u. \tag{7}$$

The substitution $u^{1+\alpha} = \sin^2\theta$ leads to the following expressions:

$$T(\alpha) = \frac{4\sqrt{2}}{\sqrt{1+\alpha}} A^{0.5(1-\alpha)} I_3(\beta) + \mathcal{O}(\varepsilon), \tag{8}$$

$$A = \left(\frac{I_1(\beta)}{I_2(\beta)}\right)^{1/2}, \tag{9}$$

$$I_1(\beta) = \int_0^{\pi/2} \cos^2\theta \sin^{1-2\beta}\theta\,\mathrm{d}\theta = 0.5B(1-\beta, 1.5) = \frac{\Gamma(1-\beta)\Gamma(1.5)}{2\Gamma(2.5-\beta)}, \tag{10}$$

$$I_2(\beta) = \int_0^{\pi/2} \cos^2\theta \sin^{5-6\beta}\theta\,\mathrm{d}\theta = 0.5B(3-3\beta, 1.5) = \frac{\Gamma(3-3\beta)\Gamma(1.5)}{2\Gamma(4.5-3\beta)}, \tag{11}$$

$$I_3(\beta) = \int_0^{\pi/2} \sin^{1-2\beta}\theta\,\mathrm{d}\theta = 0.5B(1-\beta, 0.5) = \frac{\sqrt{\pi}\,\Gamma(1-\beta)}{2\Gamma(1.5-\beta)}, \tag{12}$$

where $B(\ldots,\ldots)$ is the Beta function (see Ref. [8]), $\Gamma(\ldots)$ is the Gamma function (see Ref. [8]), with $\beta = \alpha/(1+\alpha)$.

Of course, one can use for calculations expressions (8)–(12), but sometimes it is more convenient to use approximate expressions with only elementary functions. To study the limiting cases $\alpha \ll 1$; $\alpha \gg 1$ and $\alpha \approx 1$ use will be made of these approximate expressions involving only elementary functions.

3. The case $0 < \alpha \ll 1$

For $\alpha \downarrow 0$ (so, $\beta \downarrow 0$) one has

$$\sin^{2\beta}\theta \sim 1 + 2\beta \ln(\sin\theta) + \cdots \tag{13}$$

and from Eqs. (10)–(12) (see Ref. [8])

$$I_1(0) = \tfrac{1}{3}, \quad I_2(0) = \tfrac{4}{105}, \quad I_3(0) = 1. \tag{14}$$

Expressions for I_i may be obtained as series in β:

$$I_i = I_i(0) + \beta I_i^{(1)} + \beta^2 I_i^{(2)} + \cdots, \quad i = 1, 2, 3. \tag{15}$$

Then, using expressions (13)–(15), one obtains for $\beta \downarrow 0$:

$$I_1 \sim \tfrac{1}{3} + 2\beta(\ln 2 - \tfrac{2}{3}), \tag{16}$$

$$I_2 \sim \tfrac{4}{105} + \tfrac{16}{45}\beta(\ln 2 - \tfrac{269}{140}), \tag{17}$$

$$I_3 \sim 1 + 2\beta(1 - \ln 2). \tag{18}$$

One can use Padé approximations [7] to improve the obtained result (18) for $I_3(\beta)$. A brief description of the Padé approximations is as follows. Let the function $F(\beta)$ be represented by the Maclaurin series

$$F(\beta) = \sum_{i=0}^{\infty} a_i \beta^i \quad \text{for } \beta \longrightarrow 0. \tag{19}$$

The $[m/n]$ Padé approximations are defined through the fractional rational functions $\sum_{i=0}^{m} b_i \beta^i / (1 + \sum_{i=1}^{n} c_i \beta^i)$, where the first $m + n + 1$ coefficients of the associated Maclaurin series coincide with the first terms of the series (19). In our case the [0/1] Padé approximation for I_3 (see Eq. (18)) has the form:

$$I_3 \sim \frac{1}{1 + 2\beta(\ln 2 - 1)}. \tag{20}$$

In a similar way I_1 and I_2 can be approximated:

$$I_1 \sim \frac{1}{3(1 + 2\beta(2 - 3\ln 2))}, \tag{21}$$

$$I_2 \sim \frac{4}{105(1 + \beta/3(\frac{269}{5} - 28\ln 2))}. \tag{22}$$

It is worth noting, that the expression I_1/I_2 (see Eq. (9)) has a pole at the point $\alpha \approx 0.096$, if one uses approximations (16) and (17). So, one can use these approximations only for $\alpha < 0.096$. But if one uses the Padé approximants (21) and (22), the pole of the expression I_1/I_2 occurs at the (non-physical) point $\alpha = -1.189$. It should be observed that expression (20) has a pole at the point $\alpha \approx -2.59$.

Table 1
Comparison of exact and approximate values of $I_3(\beta)$ for the case $0 < \alpha \ll 1$

α	0	1/2	2/3
Exact value (12)	1	1.19	1.39
Asymptotics (18)	1	1.16	1.25
Padé approximations (20)	1	1.18	1.33

One can estimate the accuracy of the approximations on the basis of the exact expression (12) for $I_3(\beta)$. For $0<\alpha\ll 1$ one has $1<I_3\ll\sqrt{\pi}$. Some numerical results can be found in Table 1.

Based on these approximations the period $T(\alpha)$ of the periodic solution can readily be obtained.

4. The case $\alpha\gg 1$

One can introduce the parameter $\gamma = 2/(1+\alpha)$ and suppose that $\alpha\to\infty$ (so, $\gamma\downarrow 0$). General ideas to construct asymptotic approximations may be shown by using the integral I_3:

$$I_3 = \int_0^{\pi/2} \sin^{-1+\gamma}\theta\,\mathrm{d}\theta = \int_0^{\pi/2} \theta^{-1+\gamma}\left(\frac{\sin\theta}{\theta}\right)^{-1+\gamma}\mathrm{d}\theta \sim \int_0^{\pi/2}\theta^{-1+\gamma}\frac{\mathrm{d}\theta}{(\sin\theta/\theta)(1-\gamma\ln[\sin\theta/\theta])}$$
$$\sim \int_0^{\pi/2}\theta^{-1+\gamma}\left[1+2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2^{k-1}-1)B_{2k}\theta^{2k}}{(2k)!}\right]\mathrm{d}\theta \sim \frac{1}{\gamma}\left(\frac{\pi}{2}\right)^{\gamma}\quad\text{for } \gamma\downarrow 0, \tag{23}$$

where B_{2k} are the Bernoulli numbers.

Expression (23) can be obtained by using for instance the standard formulas from Ref. [9]. Similarly,

$$I_1\sim\frac{1}{\gamma}\left(\frac{\pi}{2}\right)^{\gamma}, \tag{24}$$

$$I_2\sim\frac{1}{3\gamma}\left(\frac{\pi}{2}\right)^{3\gamma}, \tag{25}$$

$$A\sim\sqrt{3}\left(\frac{\pi}{2}\right)^{-\gamma}, \tag{26}$$

$$T\sim\frac{2\pi}{\gamma^2}3^{(\gamma-1)/2\gamma}\quad\text{for } \gamma\downarrow 0. \tag{27}$$

Now one can estimate the accuracy of the leading term of the asymptotic relations. Let us introduce the quantities:

$$\gamma I_3 = 0.5B(0.5\gamma, 0.5)\equiv A_1, \tag{28}$$
$$\gamma I_3\sim\left(\frac{\pi}{2}\right)^{\gamma}\equiv A_2. \tag{29}$$

Numerical results can be seen in Table 2, where A_1 is the exact value for γI_3, and A_2 is its asymptotic approximation as given by Eq. (29).

For $\gamma = 1$ it follows from Eq. (27) that $T = 2\pi$, and for $\gamma\to\infty$ it follows that $T\to 0$.

5. Asymptotics for $\alpha\to 1$

One can introduce the parameter $\kappa = 1-[2\alpha/(1+\alpha)]$ and suppose that $\alpha\to 1$ (so, $\kappa\to 0$). The following relation can be used:

$$\sin^{\kappa}\theta = \theta^{\kappa}\left(\frac{\sin\theta}{\theta}\right)^{\kappa}\sim\theta^{\kappa}\left(1+\kappa\ln\left(\frac{\sin\theta}{\theta}\right)\right)\quad\text{for } \kappa\to 0.$$

Table 2
Comparison of approximate and exact values of γI_3 for $\alpha\to\infty$

α	1	3	5	∞
A_1	$\pi/2$	1.30	1.20	1
A_2	$\pi/2$	1.25	1.16	1

Table 3
Comparison of approximate and exact values of I_3 for $\alpha \to 1$

α	1.5	1	0.9	0.5	0.4
Exact value of I_3	1.84	$\pi/2$	1.52	1.29	1.24
Approximate value of I_3, formula (32)	1.79	$\pi/2$	1.53	1.37	1.33

Then, for $\kappa \to 0$ it follows that

$$I_1 = \int_0^{\pi/2} \sin^{\kappa}\theta \, d\theta - \int_0^{\pi/2} \sin^{2+\kappa}\theta \, d\theta \sim -\frac{\pi}{4} + \frac{(\pi/2)^{\kappa+1}}{\kappa+1} + O(\kappa), \tag{30}$$

$$I_2 \sim \frac{\pi}{16} + O(\kappa), \tag{31}$$

$$I_3 \sim \frac{(\pi/2)^{\kappa+1}}{\kappa+1} + O(\kappa). \tag{32}$$

Some numerical results can be found in Table 3.

6. Matching of asymptotic expressions for $\alpha \gg 1$ and $\alpha \ll 1$

The reviewer of this paper proposed to construct a function to obtain asymptotics for $\alpha \gg 1$ and $\alpha \ll 1$. This very difficult problem might be a subject for another paper, but as partial solution can be proposed the following formula:

$$T = \frac{4\sqrt{2}[1 + (A-1)\alpha + A\alpha^2](1+\alpha)^2}{1+\alpha^2} \left(\frac{35 + 3\alpha}{1+\alpha}\right)^{(1-\alpha)/4}, \tag{33}$$

where $A = \pi/8\sqrt{2}$.

Formula (33) for $\alpha \to \infty$ tends to asymptotic values as given by Eq. (27), for $\alpha = 1$ it gives $T = 2\pi$, and for $\alpha = 0$ it gives the values as given by formulas (16)–(18).

7. Conclusions

The obtained asymptotic results give the possibility to use simple analytical expressions for the period of the generalized van der Pol equation for any of the values of the parameters α. More exactly:

- the asymptotics (20)–(22) can be used for $0 < \alpha < 2/3$;
- the asymptotics (30)–(32) is valid for $0.5 < \alpha < 1.5$; and
- the asymptotics (24)–(27) can be used for $\alpha > 1$.

It is worth noting, that we have overlapping domains of asymptotic validity.

Acknowledgments

This work is supported partly by the Netherlands Organization for Scientific Research (NWO). The authors thank Dr. A. Andrianov and Dr. M. Zarubinskaya for some help with symbolic computations, and preparing of manuscript. The authors also thank the anonymous reviewer for valuable suggestions.

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